

# OPTIMAL $C^{1,\alpha}$ ESTIMATES FOR A CLASS OF ELLIPTIC QUASILINEAR EQUATIONS

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**ABSTRACT.** It is well known that solutions of the following  $p$ -Laplacian type equation

$$-div(\gamma(x)|\nabla u|^{p-2}\nabla u) = f$$

are locally  $C^{1,\varepsilon}$  for some  $\varepsilon > 0$  if  $f$  and  $\gamma$  are smooth. In this article we study a much bigger class of quasilinear equations and establish a sharp interior  $C^{1,\alpha}$  estimates for weak solutions. The optimal index  $\alpha$ , which is determined by intrinsic scaling, depends on the regularity of the corresponding homogeneous equation and the integrability of  $f$ .

## 1. INTRODUCTION

Let  $\Omega$  be a bounded subset of  $\mathbb{R}^n$  ( $n \geq 2$ ) and  $u$  be a solution of

$$(1.1) \quad -div a(x, \nabla u) = f(x) \quad \text{in } \Omega,$$

in the sense of distribution. Throughout the article we shall assume  $f \in L^q(\Omega)$  for  $n < q \leq \infty$ . The continuous vector field  $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$  regular in the gradient variable  $\xi$ . In addition we assume  $a$  to satisfy

$$(1.2) \quad \begin{cases} |a(x, \xi)| + |\partial_\xi a(x, \xi)| |\xi| \leq \Lambda |\xi|^{p-1} \\ \lambda |\xi_1|^{p-2} |\xi_2|^2 \leq \langle \partial_\xi a(x, \xi_1) \xi_2, \xi_2 \rangle \\ |a(x_1, \xi) - a(x_2, \xi)| \leq \tilde{\Lambda} \omega(|x_1 - x_2|) |\xi|^{p-1} \end{cases}$$

for positive constants  $\lambda \leq \Lambda$ ,  $\tilde{\Lambda} \geq 1$ , all  $x, x_1, x_2 \in \Omega$  and  $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$ . In (1.2),  $\omega$  is a modulus of continuity in  $\mathbb{R}^+$  with  $\omega(0) = 0$ , where for some  $0 < \sigma_0 < 1$ ,  $\omega$  is Hölder continuous with exponent  $\sigma_0$ .

Clearly (1.1) is a generalization of the following standard  $p$ -Laplacian equation:

$$(1.3) \quad -div(\gamma(x)|\nabla u|^{p-2}\nabla u) = f, \quad \text{in } \Omega,$$

where  $\gamma(x)$  is a positive smooth function.

The main purpose of this article is to determine the  $C^{1,\alpha}$  estimate for solutions of (1.1) with the almost optimal  $\alpha$ . For decades people have been proving various existence and regularity results. For the simplest  $p$ -Laplacian equation

$$(1.4) \quad \Delta_p u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad \Delta_p v := div(|\nabla v|^{p-2}\nabla v),$$

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Uraltseva in [19] proved that for  $p > 1$ , weak solutions in the sense of distribution are locally  $C^{1,\alpha}$  for some  $\alpha \in (0, 1)$ , but the exact number of  $\alpha$  was not determined in her work (see also [17, 5] and the reference therein). Since then regularity estimates for quasilinear equations with varying coefficients as in (1.1) have been treated by several approaches. Here we mention two results closely related to this article: In [4, 10] Duzaar, Kussi and Mingione established the continuity of  $\nabla u$ , assuming  $f$  in the Lorentz space  $L(n, 1/(p-1))$  and  $a(\cdot, \xi)$  being Dini continuous. Moreover if  $a$  and  $f$  satisfy

$$(1.5) \quad a(\cdot, \xi) \in C^{0,\sigma} \quad \text{and} \quad f \in L^q(\Omega)$$

with  $n < q \leq \infty$  and  $0 < \sigma < 1$ , a modulus of continuity of  $\nabla u$  was established. It was discovered by Teixeira [14] that for equation (1.1), solutions are surprisingly smooth around critical points of  $u$ , even though the overall regularity of  $u$  cannot be expected due to the low regularity of the coefficients.

In a different direction, using tools from complex analysis in [8], Lindgren and Lindqvist obtained the optimal interior  $C^{1,\alpha}$  estimate for  $p$ -Laplacian equations under two major assumptions: 1: the equation is defined in two dimensional spaces and 2:  $p \geq 2$ .

Before we state our main result we introduce some important indexes that play an important role for the regularity of solutions of the equation (1.1). In relation to constant coefficient equation:

$$(1.6) \quad -\operatorname{div} a(\nabla u) = 0, \quad \text{i.e. } \omega \equiv 0$$

we use  $0 < \alpha_M \leq 1$ , depending only on  $n, p, \lambda, \Lambda$ , to denote the best  $C_{loc}^{1,\alpha}$ -regularity exponent for solutions of (1.6), see for instance [3, 13]. Also, let us denote

$$\alpha_{p,q} := \begin{cases} \min \left\{ \sigma_0, 1 - \frac{n}{q} \right\} \cdot \min \left\{ 1, \frac{1}{p-1} \right\} & \text{if } n < q < \infty, \\ \sigma_0 \cdot \min \left\{ 1, \frac{1}{p-1} \right\} & \text{if } q = \infty. \end{cases}$$

Our main result can be stated as follows:

**Theorem 1.1.** *Let  $u \in W^{1,p}(\Omega)$  be a weak solution of (1.1), where  $f \in L^q(\Omega)$  for some  $n < q \leq \infty$  and the vector field  $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying the conditions (1.2) for some  $p > 2 - \frac{1}{n}$ . Then  $u$  is locally  $C^{1,\gamma}(\Omega)$ , where*

$$(1.7) \quad \gamma = \min \{ \alpha_M^-, \alpha_{p,q} \}$$

Moreover, for any  $K \Subset \Omega$ , there holds

$$(1.8) \quad \sup_{x,y \in K, x \neq y} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\gamma} \leq C_\gamma,$$

for some constant  $C_\gamma > 0$  depending only on  $K, \Omega, n, p, \sigma_0, \lambda, \Lambda, \|f\|_{L^q(\Omega)}, \|u\|_{L^p(\Omega)}$  and  $(\alpha_M - \gamma)^{-1}$ .

Hereafter in this paper we denote  $\gamma$  as in (1.7) with  $\alpha_M^-$  denoting any positive number  $\alpha < \alpha_M$ . So the constant  $C_\gamma \rightarrow \infty$  if  $\gamma \rightarrow \alpha_M$ .

The sharpness of the index  $\gamma$  can be observed from the following examples of the  $p$ -Laplacian equation with  $p \geq 2$ :

$$\Delta_p v = f, \quad p \geq 2.$$

Clearly in this special case  $\sigma_0 = 1$  and

$$\alpha_{p,q} = \begin{cases} (1 - \frac{n}{q})/(p-1), & \text{if } q \in (n, \infty), \\ 1/(p-1), & \text{if } q = \infty. \end{cases}$$

If  $f \equiv 0$ , we should have  $\alpha \leq \alpha_M$ . If  $f$  is a positive constant, direct computation shows that

$$u(x) = \frac{p-1}{p} |x|^{p/(p-1)} \quad \text{solves} \quad \Delta_p u = n.$$

Thus  $\alpha \leq 1/(p-1)$ . Finally, if  $f$  is only assumed to be in  $L^q$  for some  $q > n$ ,

$$\tilde{u}(x) = |x|^{1+a} \quad \text{solves} \quad \Delta_p \tilde{u} = (1+a)^{p-1} a(p-1) |x|^{ap-a-1}.$$

It is easy to see that  $\Delta_p \tilde{u} \in L^q(B_1)$  and  $u \in C^{1+a}$  if  $a = \frac{1-n/q}{p-1} + \varepsilon$  for some  $\varepsilon > 0$  small. Thus if we only assume  $f \in L^q$  ( $q > n$ ) the optimal  $\alpha \leq (1-n/q)/(p-1)$ . From these examples we see that even for the  $p$ -Laplacian equation with  $p \geq 2$ , Theorem 1.1 gives the optimal  $\alpha$  as the minimum of the three indexes mentioned above.

Here we also compare Theorem 1.1 with the result of Lindgren and Lindqvist [8] for  $p$ -Laplacian equation defined in two dimensional spaces. They proved that if  $p \geq 2$  and  $f \in L^q$  for some  $2 < q < \infty$ , solutions are  $C^{1,\alpha}$  for the best  $\alpha$  possible:  $\frac{1-2/q}{p-1}$ . If  $p \geq 2$  and  $f \in L^\infty$ ,  $\alpha$  can be any positive number less than  $1/(p-1)$ . Recently, under a new oscillation estimate developed in [1], the first author, Teixeira and Urbano have shown that solutions of  $\Delta_p u = f \in L^\infty$  ( $p \geq 2$ ) are in fact locally  $C^{1,1/(p-1)}$  in the plane. Thus Theorem 1.1 is also an extension of Lindgren-Lindqvist's result for this special case because the restriction on dimension is removed.

In addition to the three restrictions mentioned above, the Hölder continuity of  $\omega$  in (1.2) certainly plays a role. Here we note that the Hölder continuity of  $\omega$  in (1.2) is definitely needed because even for the linear case

$$-\operatorname{div}(a_{ij}(x)\nabla u) = 0$$

we need to assume  $a_{ij} \in C^\sigma$  in order to obtain the  $C^{1,\sigma}$  estimate of  $u$ . Also, if the right hand side function  $f$  is only in  $L^n$  we cannot expect to have  $L^\infty$  estimate for  $\nabla u$ . See Corollary 1.1 below for the optimal assumption in this respect.

In Theorem 1.1 the assumption  $p > 2 - \frac{1}{n}$  is generally required because for such a general class of equations the compactness in  $C^1$ -topology may not hold if  $p$  is too close to 1. We refer the readers to [4, 10] for more detailed discussion of this issue. However for the model case (1.3),  $C^1$ -estimates hold for all  $p > 1$ , and so the estimate (1.8) is also valid for (1.3).

If  $q$  is slightly greater than  $n$ ,  $\gamma = (1 - n/q) \min(1, 1/(p-1))$ . The case  $q = n$  is a particularly interesting borderline case. It is well known from the work of Manfredi [12, 13] that even for the model equation (1.3), if  $f \in L^{n+\varepsilon}$  and  $\gamma \in C^{0,\varepsilon}$ ,

$\nabla u \in C^{\alpha(\varepsilon)}$  where  $\alpha(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . No explicit expression of  $\alpha(\varepsilon)$  was given in [12, 13]. The following immediate consequence of Theorem 1.1 reveals the precise information of  $\alpha(\varepsilon)$  for the more general class of equations:

**Corollary 1.1.** *Let  $u$  be a weak solution to (1.1) satisfying (1.2) for  $p \geq 2$  with*

$$\omega \in C^{0,\varepsilon} \quad \text{and} \quad f \in L^{n+\varepsilon}$$

*for some small  $\varepsilon > 0$ . Then  $\nabla u$  is locally of class  $C^{0,\alpha(\varepsilon)}$  for*

$$\alpha(\varepsilon) = \frac{\varepsilon}{n+\varepsilon} \cdot \frac{1}{p-1}.$$

Because of the generality of (1.1) some classical tools such as Schauder estimate are not readily available. Even for the case  $p = 2$ , the nonlinear vector field  $a$  satisfying (1.2) has a quadratic growth and the classical regularity estimates for elliptic equations of divergent form cannot be applied directly. We shall establish this Schauder type estimate in the second main result:

**Theorem 1.2.** *Let  $u \in H^1(\Omega)$  be a solution to (1.1) satisfying the conditions of Theorem 1.1 with  $p=2$ . Then  $u$  is locally  $C^{1,\beta}$ , where the estimate (1.8) is valid for the following exponent:*

$$\beta = \begin{cases} \min\{\sigma_0, 1 - n/q\} & \text{if } n < q < \infty \\ \sigma_0 & \text{if } q = \infty. \end{cases}$$

Here we mention the essential ingredients in the proof of the main result. For a generic solution of (1.1) we consider an appropriate neighborhood of the critical set  $\mathcal{C}(u) := \{x : \nabla u(x) = 0\}$  in small balls with radius  $r$  where  $|\nabla u| \lesssim r$ . It allows us to apply the oscillation estimates developed by the first author, Teixeira and Urbano [1], see Section 3. However, for balls of radius  $r \lesssim |\nabla u|$  the gradient becomes large and the vector field  $a(x, \xi)$  exhibits a nonlinear quadratic growth. In this case Theorem 1.2 (to be established in the Section 2) essentially provides the optimal regularity in this region. By combining estimates in these regions carefully we obtain the desired local estimates.

**Notations.** For  $B_r(x) \subset \mathbb{R}^n$ , we denote the open ball with radius  $r > 0$  centered at  $x \in \mathbb{R}^n$ , in particular we set  $B_r(0) \equiv B_r$ . If we do not mention the explicit dependence, constants depend on the usual parameters:  $n, p, \Lambda, \lambda, \tilde{\Lambda}, \|u\|_{L^p}, \|\omega\|_{C^{\sigma_0}}, \|f\|_{L^q}$  and  $\text{dist}(K, \partial\Omega)$ , for a compact set  $K \Subset \Omega$ , where  $\text{dist}(K, \partial\Omega)$  means the euclidean distance between  $K$  and the boundary of the domain  $\Omega$ .

**Organization of the paper.** In Section 2 we treat regularity results for equations with quadratic growth by providing the proof of Theorem 1.2. In Section 3, considering the general  $p$ -growth case, we derive regularity estimates for large radius and so, by using Theorem 1.2 for the small radius case, standard arguments are employed to prove Theorem 1.1.

## 2. REGULARITY ESTIMATES FOR QUASILINEAR EQUATIONS WITH QUADRATIC GROWTH

In this section we establish optimal regularity estimates for equations (1.1) with quadratic growth, i.e., the vector field  $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the conditions (1.2) for  $p = 2$ . As commented before, even for the case  $p = 2$ , the Schauder estimate has not been established before, because the vector field  $a$  has a nonlinear behavior. In this section we establish this Schauder estimate, which is Theorem 1.2, and we shall use Theorem 1.2 as a major tool to prove the optimal regularity for equations with general  $p$ -growth in Section 3.

Hereafter in this section, for  $R > 0$  and  $x_0 \in B_{1/2}$ , we consider the function  $h \in H^1(B_R(x_0))$  a solution to the following equation

$$(2.1) \quad -\operatorname{div} a(x_0, \nabla h) = 0 \quad \text{in } B_R(x_0),$$

with  $a$  satisfying the conditions in (1.2) for  $p = 2$ . Under such assumptions, we recall  $h \in W_{loc}^{2,2}(B_R(x_0))$  and so each component  $h_i := \nabla_{e_i} h$ , for  $i = 1, \dots, n$ , solves an equation of divergence form

$$(2.2) \quad -\operatorname{div}(A(x) \nabla h_i) = 0 \quad \text{for } A_{ij}(x) := \partial_{\xi_j} a^i(x_0, \nabla h(x)),$$

since  $a(x_0, \cdot)$  is a constant coefficient equation. Also, by using  $C^1$ -regularity for  $h$  and (1.2) respectively, we have  $A_{ij} \in C^0(\overline{B_R(x_0)})$  satisfying the following ellipticity condition:

$$\lambda |\xi|^2 \leq A_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2,$$

for any  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ . Therefore, by standard estimates for elliptic equations of divergence form, see for instance [6, Section 1.5], for each  $0 < \beta < 1$  there exists a universal constant  $C_\beta > 0$  such that

$$\int_{B_r(x_0)} |\nabla h|^2 dx \leq C_\beta \left(\frac{r}{R}\right)^{n\beta} \int_{B_R(x_0)} |\nabla h|^2 dx,$$

for any  $0 < r \leq R$ . Moreover, by using Poincaré's inequality for the estimate above, we obtain the following integral oscillation decay for harmonic functions:

**Lemma 2.1.** *Let  $h \in H^1(B_R(x_0))$  be a solution to (2.1) in  $B_R(x_0)$ . For each number  $0 < \alpha < 1$  there holds*

$$(2.3) \quad \int_{B_r(x_0)} |\nabla h - (\nabla h)_{x_0, r}|^2 dx \leq C_\alpha \left(\frac{r}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla h - (\nabla h)_{x_0, R}|^2 dx,$$

for any  $0 < r \leq R$  and some universal constant  $C_\alpha > 0$ .

Here we are using the classical average notation

$$(f)_{x, r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f dx.$$

Next, we prove an immediate consequence of the previous Lemma.

**Lemma 2.2.** *Let  $h \in H^1(B_R(x_0))$  be a solution to (2.1) in  $B_R(x_0)$ . For each  $0 < \alpha < 1$  there exists a universal constant  $C > 0$  such that*

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx &\leq C \left(\frac{r}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx \\ &\quad + C \int_{B_R(x_0)} |\nabla u - \nabla h|^2 dx \end{aligned}$$

for any  $u \in H^1(B_R(x_0))$  and  $0 < r \leq R$ .

*Proof.* Let us consider  $v := u - h$ . A direct computation gives us

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx &\leq C \left( \int_{B_r(x_0)} |\nabla u - (\nabla h)_{x_0,r}|^2 dx + \int_{B_r(x_0)} |\nabla v|^2 dx \right) \\ &\leq C \left( \int_{B_r(x_0)} |\nabla h - (\nabla h)_{x_0,r}|^2 dx + \int_{B_r(x_0)} |\nabla v|^2 dx \right) \end{aligned}$$

for any  $0 < r \leq R$ . Therefore, by (2.3) we conclude that

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx &\leq C \left(\frac{r}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla h - (\nabla h)_{x_0,R}|^2 dx \\ &\quad + C \int_{B_R(x_0)} |\nabla v|^2 dx \end{aligned}$$

and so

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx &\leq C \left(\frac{r}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx \\ &\quad + C \left(1 + \left(\frac{r}{R}\right)^{n+2\alpha}\right) \int_{B_R(x_0)} |\nabla v|^2 dx. \end{aligned}$$

□

In order to prove Theorem 1.2, we need the following technical Lemma. For more details see for instance [6, Lemma 3.4].

**Lemma 2.3.** *Let  $\Phi \geq 0$  be a nondecreasing function satisfying*

$$\Phi(\rho) \leq A \left(\frac{\rho}{r}\right)^\alpha \Phi(r) + Br^\beta$$

for any  $0 < \rho \leq r \leq R$  for nonnegative parameters  $R, A, B, \alpha$  and  $\beta$  for  $\alpha > \beta$ . Then for any  $\theta \in (\beta, \alpha)$  there holds

$$\Phi(\rho) \leq D \left( \left(\frac{\rho}{r}\right)^\theta \Phi(r) + Br^\beta \right)$$

for any  $0 < \rho \leq r \leq R$ , where  $D$  is a positive constant depending only on  $A, R, \beta, \theta$ .

Now, we are ready to prove the main result of this section.

*Proof of Theorem 1.2.* Let us consider  $h$  the unique solution to (2.1) satisfying

$$u - h \in H_0^1(B_R(x_0)).$$

By Mean value theorem, we observe that for each  $x \in B_r(x_0)$  there exists a real number  $s_x \in (0, 1)$  such that

$$\partial_\xi a(x_0, s_x \nabla u(x) + (1 - s_x) \nabla h(x)) \cdot (\nabla u(x) - \nabla h(x)) = a(x_0, \nabla u(x)) - a(x_0, \nabla h(x)).$$

Hence, the function  $v := u - h$  satisfies the following integral equation

$$(2.4) \quad \begin{aligned} & \int_{B_r(x_0)} \partial_\xi a(x_0, s_x \nabla u(x) + (1 - s_x) \nabla h(x)) \nabla v \cdot \nabla \phi \, dx \\ &= \int_{B_r(x_0)} (a(x_0, \nabla u(x)) - a(x_0, \nabla h(x))) \cdot \nabla \phi \, dx + \int_{B_r(x_0)} f \phi \, dx \end{aligned}$$

for any  $\phi \in H_0^1(B_r(x_0))$ . Also by (2.1),  $h$  satisfies

$$(2.5) \quad \int_{B_r(x_0)} a(x_0, \nabla h(x)) \cdot \nabla \phi \, dx = 0 \quad \forall \phi \in H_0^1(B_r(x_0)).$$

On the other hand, considering the test function  $\phi = v$ , as well as by condition (1.2) for  $p = 2$ , we obtain the following estimate

$$(2.6) \quad \lambda \int_{B_r(x_0)} |\nabla v(x)|^2 \, dx \leq \int_{B_r(x_0)} \partial_\xi a(x_0, s_x \nabla u(x) + (1 - s_x) \nabla h(x)) \nabla v \cdot \nabla v \, dx.$$

By using the estimates (2.5) and (2.6) in (2.4), we get

$$(2.7) \quad \int_{B_r(x_0)} |\nabla v|^2 \, dx \leq C \left( \omega(r)^2 \int_{B_r(x_0)} |\nabla u|^2 \, dx + \left( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} \right)$$

for some universal constant  $C > 0$ . Also, by Hölder inequality we derive

$$(2.8) \quad \left( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} \leq \left( \int_{B_r(x_0)} |f|^q \, dx \right)^{\frac{2}{q}} \cdot r^{n+2(1-n/q)}$$

for all  $q > n$ . Then, by (2.7) and (2.8) we get

$$(2.9) \quad \int_{B_r(x_0)} |\nabla u - \nabla h|^2 \, dx \leq C \left( \omega(r)^2 \int_{B_r(x_0)} |\nabla u|^2 \, dx + \|f\|_{L^q(B_1)} \cdot r^{n+2(1-n/q)} \right).$$

Hence by Lemma 2.2, for each  $0 < \alpha < 1$  there holds

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0, r}|^2 \, dx \leq C \left( \left( \frac{r}{R} \right)^{n+2\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0, R}|^2 \, dx + r^{n+2\beta} \right)$$

for any  $0 < r \leq R$  where  $\beta := \min\{\sigma_0, 1 - n/q\}$ . Therefore by Lemma 2.3, for some universal small number  $R_0 > 0$  such that for all  $0 < R \leq R_0$  there holds

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0, r}|^2 \, dx \leq C \left( \left( \frac{r}{R} \right)^{n+2\beta} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0, R}|^2 \, dx + r^{n+2\beta} \right)$$

for any  $0 < r \leq R$ . In particular, for  $R = R_0$  and  $0 < r \leq R'_0 := 2^{-\frac{1}{n+2\beta}} \cdot R$  we obtain

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0, r}|^2 \, dx \leq C \cdot r^{n+2\beta}$$

for any  $0 < r \leq R'_0$ . Finally by Campanato's embedding Theorem (see for instance [11]), we derive the desired Hölder continuity. The proof of Theorem 1.2 is complete.  $\square$

### 3. REGULARITY ESTIMATES FOR QUASILINEAR EQUATIONS WITH $p$ -GROWTH

In this section we shall consider parameters  $2 - 1/n < p$  and  $n < q \leq \infty$ . We remark that by regularity estimates for quasilinear equations under the weakest assumptions (see [17, 4, 10]), solutions to (1.1) are locally  $C^1$ , depending especially on  $\|u\|_{L^p}$ , so such solution can be defined at each point.

Theorem 1.1 follows from the following theorem by a standard argument.

**Theorem 3.1.** *Let  $u \in W^{1,p}(\Omega)$  be a weak solution to (1.1) in  $\Omega$ . For each compact set  $K \Subset \Omega$ , there exist positive constants  $C$  and  $\rho_0$  such that, for each  $0 < \rho < \rho_0$  and  $x_0 \in K$ , there holds*

$$(3.1) \quad \sup_{B_\rho(x_0)} |u(x) - u(x_0) - \nabla u(x_0) \cdot x| \leq C\rho^{\gamma+1},$$

where  $\gamma$  is determined by (1.7).

**3.1. Optimal growth estimates for large radius.** Here we derive the main novelty of these arguments: to provide for weak solutions  $u$  of (1.1) the following upper optimal control

$$(3.2) \quad \sup_{B_\rho(x_0)} |u(x) - u(x_0) + \nabla u(x_0) \cdot x| \lesssim \rho^{1+\gamma}$$

for radii

$$|\nabla u(x_0)| \lesssim \rho^\gamma.$$

**Proposition 3.1.** *Let  $u \in W^{1,p}(B_1)$  be a weak solution of (1.1) in  $B_1$ . There exist positive constants  $K_0, \rho_0$  and  $C$  such that, if*

$$(3.3) \quad |\nabla u(0)| \leq \frac{\rho^\gamma}{K_0},$$

for some  $0 < \rho < \rho_0$ , then

$$(3.4) \quad \sup_{B_\rho} |u(x) - u(0)| \leq C\rho^{\gamma+1}.$$

In order to derive the Proposition 3.1 we show, under a certain smallness regime adopted to the parameters related to equation (1.1), that  $u$  can be approximated by a solution of a constant coefficient equation as in (1.6).

**Lemma 3.1.** *Let  $u \in W^{1,p}(B_1)$  be a weak solution of (1.1) in  $B_1$ . Given  $\sigma, \kappa > 0$  there exists  $\eta > 0$  depending on  $\sigma$  and  $\kappa$ , such that, for*

$$(3.5) \quad \|u\|_{L^\infty(B_1)} \leq 1, \quad \|f\|_{L^q(B_1)} \leq \eta \quad \text{and} \quad \sup_{B_1} |a(x, \xi) - a(0, \xi)| \leq \eta |\xi|^{p-1}$$

there exists a function  $h$  in  $B_{3/4}$ , solution of some constant coefficient equation,

$$(3.6) \quad -\operatorname{div}(a_0(\nabla h)) = 0 \quad \text{in} \quad B_{3/4}$$



such that

$$(3.7) \quad \sup_{B_{1/2}} |u - h| \leq \sigma^{1+\gamma} \quad \text{and} \quad |\nabla u(0) - \nabla h(0)| \leq \kappa^\gamma.$$

*Proof.* By contradiction there exist sequences  $\{u_j\}$ ,  $\{a_j\}$  and  $\{f_j\}$ , for  $j \in \mathbb{N}$  satisfying

$$(3.8) \quad \operatorname{div}(a_j(x, \nabla u_j)) = f_j \quad \text{in } B_1$$

where

$$(3.9) \quad \|u_j\|_{L^\infty(B_1)} \leq 1, \quad \|f_j\|_{L^q(B_1)} = o(j) \quad \text{and} \quad |a_j(x, \xi) - a_j(0, \xi)| \cdot |\xi|^{1-p} = o(j),$$

but, for some positive constants  $\sigma_*, k_* > 0$ , there holds

$$(3.10) \quad \sup_{B_{1/2}} |u_j - h| > \sigma_*^{\gamma+1} \quad \text{or} \quad |\nabla h(0) - \nabla u_j(0)| > \kappa_*^\gamma$$

for any solution  $h$  in  $B_{3/4}$  of the constant coefficients equation (3.6). However, a standard regularity result for solutions of (3.8) assures that  $\{u_j\}$  is a pre-compact sequence in the  $C^1$ -topology, see [4, 10] and [14, Theorem 2.1]. Therefore  $\{u_j\}$  converges to  $u_\infty$  in  $C^1$  norm along a subsequence and  $u_\infty$  satisfies

$$\sup_{B_{1/2}} |u_j - u_\infty| \rightarrow 0 \quad \text{and} \quad |\nabla u_j(0) - \nabla u_\infty(0)| \rightarrow 0.$$

On the other hand, thanks to  $C^1$ -compactness, we can consider a universal constant  $K > 0$  such that  $|\nabla u_j| \leq K/2$  in  $B_{3/4}$ . Now, let us define

$$b_j(x, \xi) := a_j(x, \xi) \chi_{\{|\xi| \leq K\}} + K \chi_{\{|\xi| > K\}}.$$

We note that the sequence  $\{b_j(0, \cdot)\}$  is bounded and equicontinuous, therefore by Ascoli-Arzelá,  $b_j(0, \cdot) \rightarrow b_\infty(0, \cdot)$  uniformly in  $B_{1/2}$ . Hence, by (3.9) we obtain

$$|a_j(x, \xi) - \tilde{a}(0, \xi)| \leq |a_j(x, \xi) - a_j(0, \xi)| + |a_j(0, \xi) - a_\infty(0, \xi)| = o(j)$$

for any  $x \in B_{1/2}$  and  $\xi \in B_K$ . This means that  $a_j \rightarrow a_\infty$  uniformly in  $B_{1/2} \times B_K$ . By the standard arguments, we see that  $u_\infty$  solves the constant coefficients equation

$$-\operatorname{div}(a_\infty(0, \nabla u_\infty)) = 0 \quad \text{in } B_{1/2}$$

which leads to a contradiction to (3.10).  $\square$

**Proposition 3.2.** *Let  $u \in W^{1,p}(B_1)$  be a weak solution of (1.1) in  $B_1$ . There exist positive numbers  $\eta_0, \rho_0 < 1, K_0$  and  $C$  such that if*

$$\|u\|_{L^\infty(B_1)} \leq 1, \quad \|f\|_{L^q(B_1)} \leq \eta_0, \quad \sup_{B_1} |a(x, \xi) - a(0, \xi)| \leq \eta_0 |\xi|^{p-1}$$

and

$$|\nabla u(0)| \leq \frac{\rho_0^\gamma}{K_0},$$

then there exists a universal constant  $\mu$  such that

$$\sup_{B_{\rho_0}} |u(x) - \mu| \leq \rho_0^{\gamma+1}.$$

*Proof.* From Lemma 3.1 we know that for  $\sigma$  and  $\kappa$  small  $u$  is close to a solution  $h$  of some constant coefficient equation. We shall determine  $\sigma$  and  $\kappa$  later. From the result of Lemma 3.1 we have

$$(3.11) \quad \sup_{B_\rho} |u - h(0)| \leq \sup_{B_{1/2}} |u - h| + \sup_{B_\rho} |h - h(0)|, \quad \rho \leq \rho_0.$$

On the other hand, by the local regularity estimates to constant coefficients equations as in (1.6), we get

$$(3.12) \quad \sup_{B_\rho} |h - h(0)| \leq C(n, p) \rho^{\alpha_M+1} + |\nabla h(0)| \rho.$$

Here we emphasize that  $C$  depends only on  $n$  and  $p$ . As a consequence of (3.11) and (3.12) we have

$$(3.13) \quad \sup_{B_{\rho_0}} |u - h(0)| \leq \sigma^{\gamma+1} + C \rho_0^{\alpha_M+1} + (\kappa^\gamma + |\nabla u(0)|) \rho_0.$$

Setting

$$\sigma = \rho_0 \left( \frac{1}{3} \right)^{1/(\gamma+1)} \quad \text{for} \quad \rho_0 = \left( \frac{1}{3C} \right)^{1/(\alpha_M-\gamma)}$$

and

$$\kappa = \left( \frac{1}{3} - \frac{1}{K_0} \right)^{\frac{1}{\gamma}} \rho_0 \quad \text{with} \quad K_0 > 3$$

we get

$$(3.14) \quad \sup_{B_{\rho_0}} |u - \mu| \leq \rho_0^{\gamma+1},$$

where  $\mu = h(0)$  is universally bounded.  $\square$

The proof of the following Proposition makes iterative use of Proposition 3.2 showing a discrete version of Proposition 3.1.

**Proposition 3.3.** *Let  $\eta_0$  and  $K_0$  be determined as in Proposition 3.2. If a weak solution  $u \in W^{1,p}(B_1)$  of (1.1) in  $B_1$  satisfies*

$$(3.15) \quad \|u\|_{L^\infty(B_1)} \leq 1, \quad \|f\|_{L^q(B_1)} \leq \eta_0, \quad \sup_{B_1} |a(x, \xi) - a(0, \xi)| \leq \eta_0 |\xi|^{p-1}$$

and

$$|\nabla u(0)| \leq \frac{\rho_0^{k\gamma}}{K_0},$$

for some positive integer  $k \in \mathbb{N}$ , then there holds

$$\sup_{B_{\rho_0^k}} |u(x) - \mu_k| \leq \rho_0^{k(\gamma+1)},$$

where the sequence  $\{\mu_k\}_{k \in \mathbb{N}}$  satisfies

$$(3.16) \quad |\mu_{k+1} - \mu_k| \leq C \cdot \rho_0^{k(\gamma+1)}$$

for some constant  $C > 0$ .

*Proof.* The proof of Proposition 3.3 is by induction. The case  $k = 1$  follows by Proposition 3.2. Next we assume that the conclusion holds for  $k \leq i$ . Suppose  $u$  satisfies

$$|\nabla u(0)| \leq \frac{\rho_0^{(i+1)\gamma}}{K_0}.$$

Set  $\omega_i(x) := \frac{u(\rho_0^i x) - \mu_i}{\rho_0^{i(\gamma+1)}}$ . Direct computation shows that  $\omega$  satisfies

$$-\operatorname{div} a_i(x, \nabla \omega_i(x)) = f_i(x) \quad \text{in } B_1$$

for

$$a_i(x, \xi) := \rho_0^{i\gamma(1-p)} a(\rho_0^i x, \rho_0^{i\gamma} \xi)$$

satisfying the same conditions as in (1.2), and also the smallness conditions as in (3.15). Moreover,

$$f_i(x) := \rho_0^{i(\gamma(1-p)+1)} f(\rho_0^i x),$$

where it is easy to verify that

$$\|f_i\|_{L^q(B_1)} \leq \rho_0^{i(\gamma(1-p)+1-\frac{n}{q})} \|f\|_{L^q(\Omega)}$$

and so, by  $\gamma \leq \frac{1-n/q}{p-1}$  we see that  $\|f_i\|_{L^q(B_1)} \leq \eta_0$ . In addition we have

$$|\nabla \omega(0)| \leq \frac{\rho_0^\gamma}{K_0}.$$

Thus,  $\omega$  satisfies the hypotheses of Proposition 3.2 and we have

$$\sup_{B_{\rho_0}} |\omega(x) - \mu_*| \leq \rho_0^{\gamma+1}$$

for some bounded constant  $\mu_*$ . Finally, by the definition of  $\omega$  and the estimate above, we obtain

$$\sup_{y \in B_{\rho_0^{i+1}}} |u(y) - \mu_{i+1}| \leq \rho_0^{(i+1)(\gamma+1)},$$

where  $\mu_{i+1} := \mu_i + \rho_0^{i(\gamma+1)} \mu_*$  and the proof of Proposition 3.3 is complete.  $\square$

*Proof of Proposition 3.1 under smallness regime.* Let  $\rho_0$  and  $K_0$  be as in the Proposition 3.3. Given a number  $0 < \rho \leq \rho_0$ , we take an integer  $k > 0$  such that

$$\rho_0^{k+1} < \rho \leq \rho_0^k.$$

Hence, by the condition (3.3)

$$|\nabla u(0)| \leq \frac{\rho_0^{k\gamma}}{K_0},$$

and therefore, by Proposition 3.3,

$$(3.17) \quad \sup_{B_{\rho_0^k}} |u(x) - \mu_k| \leq \rho_0^{k(\gamma+1)},$$

for some sequence  $\{\mu_k\}_{k \in \mathbb{N}}$  satisfying (3.16). Setting  $\mu_* := \lim_{k \rightarrow \infty} \mu_k$ , we have

$$(3.18) \quad |\mu_k - \mu_*| \leq C' \cdot \rho_0^{k(\gamma+1)}.$$

for some constant  $C' > 0$ . Finally, using (3.17) and (3.18), we obtain

$$\sup_{B_\rho} |u(x) - \mu_*| \leq \sup_{B_{\rho_0^k}} |u(x) - \mu_k| + |\mu_k - \mu_*| \leq C \cdot \rho_0^{(k+1)(\gamma+1)} \leq C \cdot \rho^{\gamma+1}.$$

Obviously  $\mu_* = u(0)$ . Proposition 3.1 is established.  $\square$

*Universal smallness regime.* Now, we remark that the Proposition 3.1 holds for any compact set  $K \Subset \Omega$  without the smallness conditions (3.15). In fact, for  $v \in W^{1,p}(\Omega)$  as a solution of (1.1), let  $x_0 \in K$ . For positive numbers  $A_0$  and  $B_0$ , the following function:

$$u(x) := \frac{v(x_0 + A_0 x)}{B_0}$$

solves

$$-\operatorname{div} a_0(x, \nabla u) = f_0 \quad \text{in } B_1,$$

where

$$a_0(x, \xi) := (B_0/A_0)^{1-p} a(x_0 + A_0 x, B_0/A_0 \cdot \xi)$$

and

$$\|f_0\|_{L^q(B_1)} \leq B_0^{1-p} A_0^{p-n/q} \|f\|_{L^q(B_1)}.$$

Also, it is easy to see that  $a_0$  satisfies the structural condition (1.2) with

$$|a_0(x, \xi) - a_0(0, \xi)| \leq \tilde{\Lambda} \omega(A_0 |x|).$$

Therefore, by choosing

$$A_0 := \min \left\{ 1, \operatorname{dist}(K, \partial\Omega)/2, \omega^{-1}(\tilde{\Lambda} \cdot \eta_0) / \operatorname{dist}(K, \partial\Omega) \right\}$$

and

$$B_0 := \max \left\{ 1, \|v\|_{L^\infty(\Omega)}, \sqrt[p-1]{\|f\|_{L^q(\Omega)} \eta_0^{-1}} \right\},$$

we conclude that  $a_0$  satisfies the same structural conditions (1.2) as well as the smallness assumptions in (3.15). Moreover  $u$  is normalized, i.e.,  $\|u\|_{L^\infty(B_1)} \leq 1$ .

Also, we would like to point out that by regularity theory for quasilinear equations the upper bound for  $L^\infty$ -norm follows:  $\|v\|_{L^\infty(\Omega)} \leq C \|v\|_{L^p(\Omega)}$ , for some universal constant  $C > 0$  depending on the following parameters:  $n, p, \Lambda, \lambda, \tilde{\Lambda}, \|\omega\|_{C^{\sigma_0}}, \|f\|_{L^q}, K$  and  $\Omega$ . Therefore, the normalization constant  $B_0$  depends only on  $L^p$ -norm of  $v$  and universal parameters cited above. We are ready to prove Theorem 3.1.

**3.2. Proof of Theorem 3.1.** Under a universal normalization argument, with no loss of generality, we consider  $|\nabla u(0)|$  universally small, more precisely satisfying

$$\rho_* := [K_0 \cdot |\nabla u(0)|]^{\frac{1}{\gamma}} \leq \rho_0,$$

for the universal constant  $\rho_0 > 0$  described in Proposition 3.1. In order to show the estimate (3.1) for any  $0 < \rho \leq \rho_0$ , we shall consider two cases.

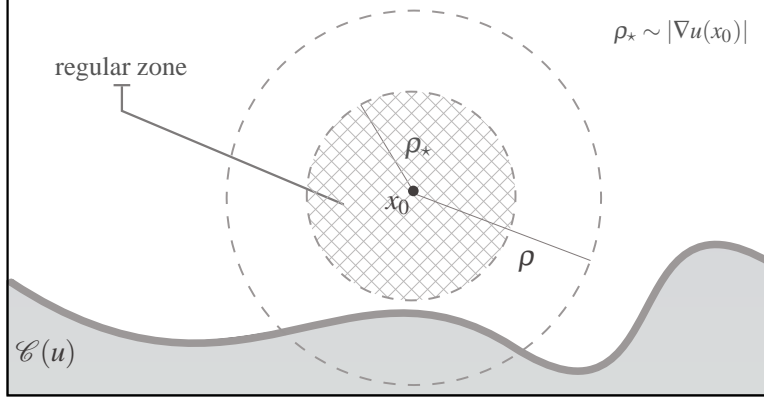


FIGURE 1. This picture represents the obtaining of the regularity estimate by considering balls with radius  $\rho$  in the following cases: the regular case  $\rho \lesssim |\nabla u(x_0)|$  as well as the large radius case  $|\nabla u(x_0)| \lesssim \rho$ .

*Case 1:*  $\rho_* \leq \rho \leq \rho_0$ . For this case, by using Proposition 3.1 we can obtain directly estimate (3.1). To conclude the proof of Theorem 3.1 we only need to consider the following case.

*Case 2:*  $0 < \rho < \rho_*$ . Initially, we define

$$v(x) := \frac{u(\rho_* x) - u(0)}{\rho_*^{\gamma+1}} \quad \text{in } B_1.$$

Direct computation gives

$$(3.19) \quad -\operatorname{div} a_*(x, \nabla v) = f_*(x) \quad \text{in } B_1,$$

where

$$a_*(x, \xi) = \rho_*^{\gamma(1-p)} a(\rho_* x, \rho_*^\gamma \xi) \quad \text{and} \quad f_*(x) := \rho_*^{\gamma(1-p)+1} f(\rho_* x).$$

Clearly  $a_*$  is under the conditions in (1.2), also

$$\|f_*\|_{L^q(B_1)} \leq \rho_*^{\gamma(1-p)+1-n/q} \|f\|_{L^q(B_1)} \leq \|f\|_{L^q(B_1)}.$$

On the other hand, by considering Proposition 3.1 for the radius  $\rho_*$ , we can find a universal  $C > 0$ , such that the following  $L^p$ -boundness holds

$$\|v\|_{L^p(B_1)} \leq \sup_{x \in B_1} |v(x)| \cdot |B_1|^{1/p} = \sup_{x \in B_{\rho_*}} \frac{|u(x) - u(0)|}{\rho_*^\gamma} \cdot |B_1|^{1/p} \leq C.$$

Consequently, by applying  $C^0$ -estimates for  $\nabla v$ , there exists  $\tau_* > 0$  such that

$$\operatorname{osc}_{B_{\tau_*}} |\nabla v| < \frac{1}{2K_0},$$

since  $|\nabla v(0)| = \frac{1}{K_0}$ , which implies  $|\nabla v(x)| > \frac{1}{2K_0}$  in  $B_{\tau_*}$ . Therefore, for some universal constant  $c_0 > 0$ , there holds

$$(3.20) \quad c_0 \leq |\nabla v| \leq c_0^{-1} \quad \text{in } B_{\tau_*}.$$

Therefore, in view of (3.20), the equation (3.19) can be considered strictly as a nonlinear partial differential equation with quadratic growth, i.e., satisfying the conditions in (1.2) for  $p = 2$  within  $B_{\tau_*}$ . Therefore, by Theorem 1.2 we obtain the following estimate

$$(3.21) \quad \sup_{B_r} |v(x) - v(0) - x \cdot \nabla v(0)| \leq C_1 r^\beta,$$

for each  $0 < r \leq \tau_*/2$ , where

$$\beta = \begin{cases} \min\{\sigma_0, 1 - \frac{n}{q}\} & \text{if } q < \infty, \\ \sigma_0, & \text{if } q = \infty. \end{cases}$$

It is easy to see from the definition of  $\beta_\varepsilon$ , that  $\beta > \gamma$  for any  $p > 1$  and  $n < q \leq \infty$ . Therefore, by (3.21) we have

$$(3.22) \quad \sup_{y \in B_{\rho_* r}} |u(y) - u(0) - y \cdot \nabla u(0)| \leq C_1 \tau_*^{\gamma+1} r^{\beta_\varepsilon+1} \leq C_1 (\rho_* r)^{\gamma+1}$$

for  $0 < r \leq \tau_*/2$ . Hence, the estimate (3.1) holds whenever

$$0 < r \leq \tau_* \rho_*/2.$$

To conclude this case (and so the proof of Theorem 3.1), we have to show that the estimate (3.22) holds for

$$\tau_* \rho_*/2 < r < \rho_*.$$

For this purpose, as the estimate (3.1) holds precisely for the radius  $\rho_*$ , we have

$$\sup_{B_r} |u(x) - u(0) - x \cdot \nabla u(0)| \leq C \rho_*^{\gamma+1} \leq C \left( \frac{2}{\tau_*} \right)^{\gamma+1} \cdot r^{\gamma+1},$$

and so Theorem (3.1) is established.

*Proof of Theorem 1.1.* Finally, we show how Theorem 1.1 follows from Theorem 3.1 by a standard argument. Let  $x, y \in B_{1/4}$  and without loss of generality we assume  $x = -re_1, y = re_1$ , where  $e_1 = (1, 0, \dots, 0)$  and  $e_i$  is understood similarly. From Theorem 3.1 we clearly have

$$v(y) = v(x) + 2\partial_1 v(x)r + O(r^\gamma)$$

and

$$v(x) = v(y) - 2\partial_1 v(y)r + O(r^\gamma).$$

The two equations above lead to

$$|\partial_1 v(x) - \partial_1 v(y)| \cdot r^{1-\gamma} \leq C.$$

For  $i = 2, \dots, n$  we fix  $z = \frac{x+y}{2} + re_i$ . Also by Theorem 3.1 we have

$$(3.23) \quad v(z) = v(x) + \nabla v(x) \cdot (z - x) + O(r^\gamma)$$

and

$$(3.24) \quad v(z) = v(y) + \nabla v(y) \cdot (z - y) + O(r^\gamma).$$

By the definition of  $z$  we have

$$\nabla v(y) \cdot (z - y) = \partial_1 v(y)(-r) + \partial_i v(y)r, \quad \nabla v(x) \cdot (z - x) = \partial_1 v(x)r + \partial_i v(x)r.$$

Using these equations, from (3.23) and (3.24), we have

$$v(y) - v(x) - \partial_1 v(y)r + (\partial_i v(y) - \partial_i v(x))r - \partial_1 v(x)r = O(r^\gamma).$$

Using

$$v(y) - v(x) = 2\partial_1 v(x)r + O(r^\gamma)$$

we have

$$\partial_i v(y) - \partial_i v(x) = O(r^{\gamma-1}), \quad i = 2, \dots, n.$$

Therefore Theorem 1.1 is established by using the estimate above on  $K \Subset \Omega$  and a standard covering argument.  $\square$

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